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Effects of Causality and Joint Conditions on Method of Reverberation-Ray Matrix

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We reexamine the approximations for amplitude vectors of steady-state waves in framed structures as reported previously by Pao et al. (Pao, Y. H., Howard, S. M., and Keh, D. C., "Dynamic Response and Wave Propagation in Plane Trusses and Frames," *AIAA Journal*, Vol. 37, No. 5, 1999, pp. 594–603). The arrival and departure wave vectors have been determined exactly in the reverberation-ray matrix from the joint condition of the structure and the phase relation between them within each member. In the approximations the departure vector is obtained by truncating an infinite power series of reverberation-ray matrix to a polynomial of N degree. The arrival vector is given by another polynomial of N degree if it is calculated from the phase relation as in a previous paper, but by polynomials of $(N-1)$ degree if calculated from the joint conditions as in this study. The newly found set of approximation is preferred to the previous one as it satisfies the causal condition as well as the joint conditions in successive orders of approximation. The differences between these two solutions are, however, very small numerically for large orders of N.

Nomenclature

$\boldsymbol{A}$	=	arrival response phase matrix
$\boldsymbol{a}$	=	exact solution of arrival wave vector
$\boldsymbol{a}^{(N)}$	=	Nth-order approximation of the arrival wave vector
$\boldsymbol{D}$	=	departure response phase matrix
$\boldsymbol{d}$	=	exact solution of departure wave vector
$\boldsymbol{d}^{(N)}$	=	Nth-order approximation of the departure wave vector
k	=	wave number
N	=	times of the reverberation
$\boldsymbol{P}$	=	phase matrix
$\boldsymbol{R}$	=	reverberation-ray matrix
$\boldsymbol{S}$	=	scattering matrix
$\boldsymbol{s}$	=	source wave vector
$\boldsymbol{U}$	=	permutation matrix
$\boldsymbol{u}$	=	transient response for the entire structure
$\hat{\boldsymbol{u}}$	=	steady-state response for the entire structure
$\hat{\boldsymbol{u}}_A$	=	transient response with solution A
$\hat{\boldsymbol{u}}_B$	=	transient response with solution B

I. Introduction

In two previous papers[1,2] the method of reverberation-ray matrix has been proposed to analyze the dynamic responses of framed structure. The steady-state waves for the axial and flexural motion of each structural member are expressed by two sets of wave functions in different local coordinate systems. Each set is composed of an arrival wave vector $\boldsymbol{a}$ and departure wave vector $\boldsymbol{d}$, moving toward and away from the joints of the structure, respectively. The unknown amplitudes $\boldsymbol{a}$ and $\boldsymbol{d}$ are determined from two conditions: 1) the joint condition, which is a balance of forces and moments at each joint for all connected members and compatibility of the displacements at the joint, and 2) the phase relation, which consists of equations relating the arrival and departure waves in each member. From these two conditions $\boldsymbol{d}$ is determined as $[\boldsymbol{I}-\boldsymbol{R}]^{-1}\boldsymbol{s}$, where $\boldsymbol{a}$ is calculated from condition 2 for the phase relation.

To avoid the singularity in the numerical evaluation of the inverse Fourier transform for transient wave, the matrix $[\boldsymbol{I}-\boldsymbol{R}]^{-1}\boldsymbol{s}$ is expanded into a power series of $\boldsymbol{R}$ and then truncated at term of $\boldsymbol{R}^N$. The departure wave vector is then approximated by $\boldsymbol{d}^{(N)}$, a polynomial of N degree, and the arrival wave vector is approximated by $\boldsymbol{a}^{(N)}$, another polynomial of N degree. Dynamic strains over a long time period in each member of a laboratory truss were calculated and compared favorably with experimental data.[1,2]

Recently, in attempt to evaluate the deflection of a structure, the authors tried to calculate the approximate value of arrival wave vector from the joint condition 1), $\boldsymbol{d}$ being approximated by $\boldsymbol{d}^{(N)}$, and found the approximate value for $\boldsymbol{a}$ should be $\boldsymbol{a}^{(N-1)}$ (a polynomial of $N-1$ terms), instead of $\boldsymbol{a}^{(N)}$. To show the difference and to trace the possible source of discrepancy, we summarize in the next section the mathematical procedure in Ref. 2 that leads to the approximation $[\boldsymbol{a}^{(N)}, \boldsymbol{d}^{(N)}]$, called the solution A. In Sec. III we derive the solution $[\boldsymbol{a}^{(N-1)}, \boldsymbol{d}^{(N)}]$, called the solution B, and compare these two solutions step by step in the derivations.

Finally, in Sec. IV we discuss the newly found physical implication of condition 2) for the phase relation, which is actually a condition of causality,[3] that is, the effect cannot precede the cause. Based on this causal condition, we show that the departure wave vector in one set of local coordinate must precede the arrival wave vector in other local coordinate within the same member even for steady-state waves. Furthermore, we find that solution B in successive order of approximation satisfies the conditions at more and more joints away from the source points, whereas solution A does not. The solution B of each order, however, differs only slightly from the solution A of the same order for large order of approximation. This is demonstrated by the numerical examples.

II. Transient and Steady-State Response of Structure

Consider a planar structure as shown in Fig. 1 that represents the model structure used in Refs. 1 and 2. The two-dimensional frame is composed of m number $(=17)$ of structural members at which are connected at $n(=10)$ joints with pinned or rigid connectors. The structure is subject to dynamic loads of time function $f(t)$ applied at one or many joints. Let the Fourier-transformed component of axial displacement be denoted by $\hat{u}^{JK}$, that of transverse displacement be $\hat{v}^{JK}$, and that of angle of rotation be $\hat{\phi}^{JK}$ in each member with local

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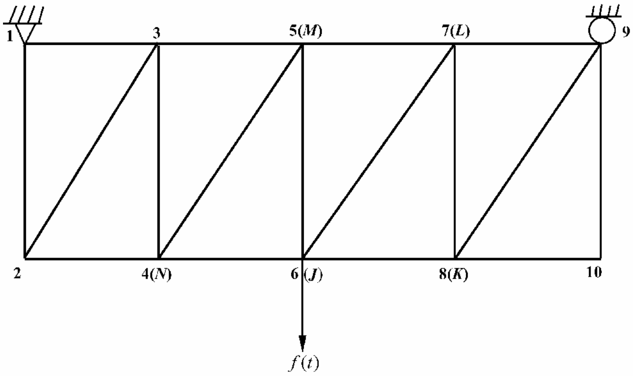

Fig. 1 Geometry of a planar structure with dynamic loading of time function $f(t)$.

axis x^{JK} from the joint J to joint K [Eq. (2) in Ref. 2]. The steady-state axial and flexural responses in the member JK, corresponding to the Fourier-transformed impulse response caused by a delta time input function, are expressed in matrix form as

$$\begin{bmatrix}\hat{u}\\ \hat{v}\\ \hat{\phi}\end{bmatrix} = \begin{bmatrix} e^{ik_1x} & 0 & 0\\ 0 & (1+\alpha_2)e^{ik_2x} & (1+\alpha_3)e^{ik_3x}\\ 0 & ik_2e^{ik_2x} & ik_3e^{ik_3x}\end{bmatrix}\begin{bmatrix}a_1\\ a_2\\ a_3\end{bmatrix} + \begin{bmatrix} e^{-ik_1x} & 0 & 0\\ 0 & (1+\alpha_2)e^{-ik_2x} & (1+\alpha_3)e^{-ik_3x}\\ 0 & -ik_2e^{-ik_2x} & -ik_3e^{-ik_3x}\end{bmatrix}\begin{bmatrix}d_1\\ d_2\\ d_3\end{bmatrix}$$

[which is derived from Eqs. (6a) and (8–10) in Ref. 2; superscripts JK are omitted]. In the matrix equation k_1^{JK} is the wave number for axial displacement, $k_{2,3}^{JK}$ are the wave numbers for the transverse displacement and the rotational angle of the flexural wave respectively, $a_{1,2,3}^{JK}$ are the unknown amplitudes of the waves arriving at the joint J, and $d_{1,2,3}^{JK}$ are those departing from the same joint J. The preceding matrix equation can be written compactly as

$$\hat{\boldsymbol{u}}^{JK}(x,\omega) = \boldsymbol{A}^{JK}(\boldsymbol{k}x)\boldsymbol{a}^{JK}(\omega) + \boldsymbol{D}^{JK}(\boldsymbol{k}x)\boldsymbol{d}^{JK}(\omega) \tag{1}$$

Corresponding to the local axis x^{KJ} from the joints K to J, there is another expression for the steady-state wave $\hat{\boldsymbol{u}}^{KJ}$ in the same member as

$$\hat{\boldsymbol{u}}^{KJ}(x,\omega) = \boldsymbol{A}^{KJ}(\boldsymbol{k}x)\boldsymbol{a}^{KJ}(\omega) + \boldsymbol{D}^{KJ}(\boldsymbol{k}x)\boldsymbol{d}^{KJ}(\omega) \tag{2}$$

Stacking up the displacement vectors and the local arrival (departure) wave vectors of all members, we obtain the global displacement vector for the entire structure in steady-state response as

$$\hat{\boldsymbol{u}}(\boldsymbol{x},\omega) = \boldsymbol{A}(\boldsymbol{kx})\boldsymbol{a}(\omega) + \boldsymbol{D}(\boldsymbol{kx})\boldsymbol{d}(\omega) \tag{3}$$

where $\boldsymbol{a}$ and $\boldsymbol{d}$ are named the global arrival and departure vectors respectively in papers.[1,2] The square matrices $\boldsymbol{A}$ and $\boldsymbol{D}$ will be called arrival and departure response phase matrices. The transient response of the structure for the loading with a causal time function $f(t)$, that is, $[f(t)=0$ for $t<0]$ is then given by

$$\boldsymbol{u}(\boldsymbol{x},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\omega)\hat{\boldsymbol{u}}(\boldsymbol{x},\omega)e^{i\omega t}\,\mathrm{d}\omega \tag{4}$$

where $\hat{f}(\omega)$ is the one-sided Fourier transform of $f(t)$.

In the solution given by Eq. (3) or Eq. (4), the unknowns $\boldsymbol{a}$ and $\boldsymbol{d}$ are determined from two conditions: 1) the joint condition of the structure and 2) the phase relation between $\boldsymbol{a}$ and $\boldsymbol{d}$ within each member. The joint conditions specify the balances of forces and moments of all members collected at the joints and compatibility conditions of all displacements at the same joints.[2] The final answer is

$$\boldsymbol{d} = \boldsymbol{Sa} + \boldsymbol{s} \tag{5}$$

[which is also Eq. (8) in Ref. 1 and Eq. (16) in Ref. 2]. $\boldsymbol{S}$ is the scattering matrix relating $\boldsymbol{a}$ to $\boldsymbol{d}$, and $\boldsymbol{s}$ is the source wave vector in all members generated by the applied loads at joints for $f(t)=\delta(t)$. This is a system of $6m$ equations for two sets of unknowns $\boldsymbol{a}$ and $\boldsymbol{d}$, each with $6m$ elements. This system must be supplemented by another set of $6m$ equations in order to determine $12m$ unknowns.

The supplementary equations are provided by the condition 2) phase relation. This condition is based on the observations of Eqs. (1) and (2) that the element $\boldsymbol{a}^{JK}$ in the column matrix of $\boldsymbol{a}$ differs from the element $\boldsymbol{d}^{KJ}$ in the column matrix of $\tilde{\boldsymbol{d}}$ only by a phase factor $-e^{-ikl}$, where ℓ is the length of the corresponding member. In global form this condition of phase correlation gives rise to another $6m$ equations of the following matrix form:

$$\begin{aligned}\boldsymbol{a} &= \boldsymbol{P}(\boldsymbol{k},\boldsymbol{l})\tilde{\boldsymbol{d}}\\ &= \boldsymbol{P}(\boldsymbol{k},\boldsymbol{l})\boldsymbol{U}\boldsymbol{d}\end{aligned} \tag{6}$$

[which are also Eqs. (13), (14) in Ref. 1 and Eq. (20) in Ref. 2], where $\boldsymbol{P}(\boldsymbol{k},\boldsymbol{l})$ is the diagonal phase matrix with elements $-e^{-ikl}$ and $\boldsymbol{U}$ is the unit permutation matrix relating the two column matrices, with $\boldsymbol{d}$ containing the elements $\boldsymbol{d}^{JK}$ and $\tilde{\boldsymbol{d}}$ containing the elements $\boldsymbol{d}^{KJ}$.

Solving the preceding two equations simultaneously, one finds the solutions for $\boldsymbol{d}$:

$$\boldsymbol{d} = (\boldsymbol{I}-\boldsymbol{R})^{-1}\boldsymbol{s} \tag{7}$$

[which are also Eq. (17) in Ref. 1 and Eq. (22) in Ref. 2], where $\boldsymbol{R}=\boldsymbol{SPU}$ is called reverberation-ray matrix for the structure. Substituting the preceding solution into Eq. (6) and the solutions for $\boldsymbol{d}$ and $\boldsymbol{a}$ so determined into Eq. (3), we obtain the steady-state solution for the entire structure:

$$\begin{aligned}\hat{\boldsymbol{u}}(\boldsymbol{x},\omega) &= [\boldsymbol{A}(\boldsymbol{k}x)\boldsymbol{PU} + \boldsymbol{D}(\boldsymbol{k}x)]\boldsymbol{d}\\ &= (\boldsymbol{APU}+\boldsymbol{D})(\boldsymbol{I}-\boldsymbol{R})^{-1}\boldsymbol{s}\end{aligned} \tag{8}$$

The transient wave solution is then obtained by carrying out the inverse Fourier transform in Eq. (4).

To avoid the singularity of the matrix $(\boldsymbol{I}-\boldsymbol{R})^{-1}$ in the inverse transform, the matrix is expanded into power series (Newman series), such as

$$\begin{aligned}\boldsymbol{d} &= (\boldsymbol{I}-\boldsymbol{R})^{-1}\boldsymbol{s} = (\boldsymbol{I}+\boldsymbol{R}+\boldsymbol{R}^2+\cdots+\boldsymbol{R}^N+\cdots)\boldsymbol{s}\\ \boldsymbol{a} &= \boldsymbol{PU}(\boldsymbol{I}-\boldsymbol{R})^{-1}\boldsymbol{s} = \boldsymbol{PU}(\boldsymbol{I}+\boldsymbol{R}+\boldsymbol{R}^2+\cdots+\boldsymbol{R}^N+\cdots)\boldsymbol{s}\end{aligned} \tag{9}$$

The integration in Eq. (4) can then be evaluated numerically term by term with the fast Fourier transform algorithm.

In the actual calculations for various dynamic responses showed in paper,[1] the infinite series is truncated to a finite sum according to the orders of approximation. Let the truncated series be represented by the polynomials of power $N (N+1$ terms), which is designated as the Nth-order approximation of the departing wave vector $\boldsymbol{d}^{(N)}$:

$$\boldsymbol{d} \approx \boldsymbol{d}^{(N)} \equiv (\boldsymbol{I}+\boldsymbol{R}+\boldsymbol{R}^2+\cdots+\boldsymbol{R}^N)\boldsymbol{s}, \qquad N=0,1,2\ldots \tag{10}$$

The arrival wave vector is then approximated by another polynomials of $N+1$ terms $\boldsymbol{a}^{(N)}$:

$$\boldsymbol{a} \approx \boldsymbol{PU}(\boldsymbol{I}+\boldsymbol{R}+\boldsymbol{R}^2+\cdots+\boldsymbol{R}^N)\boldsymbol{s} \equiv \boldsymbol{a}^{(N)}(=\boldsymbol{a}_A) \tag{11}$$

The Nth-order approximation for the steady-state displacement $\hat{\boldsymbol{u}}$ in Eq. (8) is thus given by

$$\begin{aligned}\hat{\boldsymbol{u}}_A^{(N)}(\boldsymbol{x},\omega) &= \boldsymbol{A}\boldsymbol{a}^{(N)} + \boldsymbol{D}\boldsymbol{d}^{(N)}\\ &= (\boldsymbol{APU}+\boldsymbol{D})(\boldsymbol{I}+\boldsymbol{R}+\boldsymbol{R}^2+\cdots+\boldsymbol{R}^N)\boldsymbol{s}\end{aligned} \tag{12}$$

The transient wave varied up to preset time is finally obtained by substituting $\hat{\boldsymbol{u}}_A^{(N)}$ into Eq. (4). For the purpose of comparison with the alternative solution, we designate the result of Eqs. (11) and (12) as solution A marked by the subscript A.

III. Alternative Solution

The mathematical analysis and solution up to Eq. (9) are exact, and the approximations in Eqs. (10–12) are reasonable. The solution A given by Eq. (12) formed the basis for calculating dynamic responses of planar structures.[2] As an exercise, the first author of this paper applied solution A to analyze the axial and flexural waves in a simply supported beam, generated by a concentrated force of delta function in time applied at any point in the beam. For searching a simple structure, the scattering matrix and the reverberation matrix are derived analytically, and the deflections at hinged and roller supported ends were found to be different from zero for first few orders approximation. The author then searched for an alternative solution by substituting the truncated series $\boldsymbol{d}^{(N)}$ of Eq. (10) into Eq. (5) and found that the arrival vector $\boldsymbol{a}$ should be approximated by different polynomials:

$$\boldsymbol{a} = \boldsymbol{S}^{-1}(\boldsymbol{d} - \boldsymbol{s}) \approx \boldsymbol{S}^{-1}(\boldsymbol{R} + \boldsymbol{R}^2 + \cdots + \boldsymbol{R}^N)\boldsymbol{s} \tag{13}$$

Because $\boldsymbol{S}^{-1}\boldsymbol{R} = \boldsymbol{PU}$, the preceding polynomial of N terms is defined as $\boldsymbol{a}^{(N-1)}$ in Eq. (7), and the alternative approximation is given by

$$\boldsymbol{a} \approx \boldsymbol{PU}(\boldsymbol{I} + \boldsymbol{R} + \cdots + \boldsymbol{R}^{N-1}) \equiv \boldsymbol{a}^{(N-1)} (= \boldsymbol{a}_B) \tag{14}$$

The Nth-order approximation for $\hat{\boldsymbol{u}}$ in Eq. (8) is thus changed to

$$\hat{\boldsymbol{u}}_B^{(N)}(\boldsymbol{x}, \omega) = \boldsymbol{APUa}^{(N-1)} + \boldsymbol{Dd}^{(N)} \tag{15}$$

The first term of the preceding solution differs from that of solution A in Eq. (12). We thus have inserted the subscript B to designate it as solution B.

To find an answer to the dilemma of twin solutions, we compare first the analytical expressions of Eq. (13) with those of Eq. (6) as they appear to be different. By substituting $\boldsymbol{d} = (\boldsymbol{I} - \boldsymbol{R})^{-1}\boldsymbol{s}$ in Eq. (7) into both expressions and making use the identity $(\boldsymbol{I} - \boldsymbol{R})^{-1} = \boldsymbol{R}(\boldsymbol{I} - \boldsymbol{R})^{-1}\boldsymbol{R}^{-1}$ because $\boldsymbol{R}$ and $(\boldsymbol{I} - \boldsymbol{R})$ are commutative, we find

$$\boldsymbol{a} = \boldsymbol{S}^{-1}(\boldsymbol{I} - \boldsymbol{R})^{-1}\boldsymbol{Rs} = \boldsymbol{PU}(\boldsymbol{I} - \boldsymbol{R})^{-1}\boldsymbol{s} = \boldsymbol{PUd} \tag{16}$$

The two analytical expressions, different in appearance, are actually equal.

Next, we substitute the infinite Newman series expansion of $\boldsymbol{d}$ in Eq. (9) into Eq. (5) and obtain

$$\boldsymbol{a} = \boldsymbol{S}^{-1}(\boldsymbol{R} + \boldsymbol{R}^2 + \cdots + \boldsymbol{R}^N + \cdots)\boldsymbol{s} = \boldsymbol{PU}(\boldsymbol{I} + \boldsymbol{R} + \cdots + \boldsymbol{R}^{N-1} + \cdots)\boldsymbol{s} \tag{17}$$

This expression is identical to that for $\boldsymbol{a}$ in Eq. (9). If the infinite series is truncated to Nth at this stage ($N + 1$ terms), we recover the approximation $\boldsymbol{a}^{(N)}$ of solution A.

In summary, we have employed two procedures to evaluate the wave vector $\boldsymbol{a}$ from the known solution $\boldsymbol{d}$ in Eq. (9). The first procedure is to substitute $\boldsymbol{d}$ into Eq. (6) based on condition 2 as discussed in the preceding section, and the second procedure is to substitute it into Eq. (5) based on condition 1 as discussed in this section. There is no difference between these two solutions in analytical form, or in the form of infinite Newman series. There is also no difference between the two approximate solutions if the infinite series for $\boldsymbol{d}$ and that for $\boldsymbol{a}$ are truncated to the same order simultaneously. The two approximate solutions differ, however, if the infinite series for $\boldsymbol{d}$ is truncated first to $\boldsymbol{d}^{(N)}$, which is then used to calculate the approximate value for $\boldsymbol{a}$. The remaining question is which one of these two solutions we should use in the future.

IV. Condition of Causality and Joint Conditions

In the process of tracking down the source of difference between solution A and solution B, we have come to realize that the phase relation, Eq. (6), implies a physical condition of causality, that is, the effect cannot precede the cause.[3] To see this, we return to the setting up of the wave solutions of Eqs. (1) and (2) in two different local coordinate systems. They represent the same wave in member JK generated by a loading with harmonic time function $f(t) = e^{i\omega t}(-\infty < t < \infty)$. There is no causal relation between the sinusoidal loading (source) and the harmonic wave in either coordinate (response). There is also no cause–effect relation between $\boldsymbol{a}$ and $\boldsymbol{d}$ when they are expressed in the same local coordinate system. The reason is that both arrival and departure waves are generated long before $t = 0$, and an observer at given time could not distinguish which one comes first. The global form of the steady-state waves, $\hat{\boldsymbol{u}}(\boldsymbol{x}, \omega)$ in Eq. (3), is still noncausal in time, but the transient response, $\boldsymbol{u}(\boldsymbol{x}, t)$ in Eq. (4), is causal because $f(t)$ is assumed causal. It means that the transient response (effect) cannot precede the suddenly applied loading (cause) in the structure.

As soon as the two sets of steady-state wave functions in different local coordinates are connected by Eq. (6), the sequential order for $\boldsymbol{a}$ and $\boldsymbol{d}$ is established. This is seen by restoring the phase factor to Eq. (6) for the waves in the member JK given by Eqs. (1) and (2):

$$a_q^{JK}(\omega)e^{i\omega(t + x^{JK}/c_q)} = -d_q^{KJ}(\omega)e^{-ik_q x^{KJ}}e^{i\omega t}, \qquad q = 1, 2, 3, \ldots \tag{18}$$

This means the wave arriving at section x^{JK} toward joint J in one local coordinate is the same as the wave at section x^{KJ} departing from joint K in another coordinate; the negative sign in the equation indicates two displacements at the opposite sides of the same section. Because $x^{KJ} = l^{JK} - x^{JK}$, the preceding relation is simplified to

$$\boldsymbol{a}^{JK}(\omega)e^{i\omega t} = -\boldsymbol{d}^{KJ}(\omega)e^{i\omega(t - l^{JK}/c)} \tag{19}$$

where the common factor $e^{i\omega x^{JK}/c_q}$ has been eliminated. The preceding equation states that the arrival wave $\boldsymbol{a}$ at time t is the same as the departure wave $\boldsymbol{d}$ from joint K at time $t - l^{JK}/c$, and the departure wave $\boldsymbol{d}$ in one set of coordinate precedes the arrival wave in another set. This sequential order is another form of causal condition for the departure wave vector $\tilde{\boldsymbol{d}}$ (cause) and the arrival wave vector $\boldsymbol{a}$ (effect).

Once Eq. (6) is understood to be a condition of causality, we reexamine Eq. (5) in global form, which is the stacking of local scattering equations at all joints J:

$$\boldsymbol{d}^J = \boldsymbol{S}^J\boldsymbol{a}^J + \boldsymbol{s}^J \tag{20}$$

[which is Eq. (15) in Ref. 2]. The preceding equation is in reality the consequence of applying the common boundary conditions for all wave equations in members JK, JL, … connected at the joint J. These boundary conditions, and hence Eqs. (20) or (5), must be satisfied for all waves passing through the joint at time t. Therefore, we are inclined to select Eq. (5) for calculating the amplitude $\boldsymbol{a}$ from the amplitude $\boldsymbol{d}$ as in the second procedure according to the causal condition. We note, however, that the result based on the selection of Eq. (6) as in first procedure is still correct so long as the exact matrix solutions are used. We thus proceed further to examine in detail the approximate solutions of the two procedures.

To make the final choice, we compare first the lower-order approximations of both. From Eq. (12) we find the approximate solutions in successive orders of solution A:

$$\hat{\boldsymbol{u}}_A^{(0)} = \boldsymbol{Aa}^{(0)} + \boldsymbol{Dd}^{(0)}$$
$$\hat{\boldsymbol{u}}_A^{(1)} = \boldsymbol{Aa}^{(1)} + \boldsymbol{Dd}^{(1)}$$
$$\hat{\boldsymbol{u}}_A^{(2)} = \boldsymbol{Aa}^{(2)} + \boldsymbol{Dd}^{(2)}$$
$$\ldots \tag{21}$$

The corresponding orders of approximation of solution B as given in Eq. (15) are

$$\hat{\boldsymbol{u}}_B^{(0)} = \boldsymbol{0} + \boldsymbol{D}\boldsymbol{d}^{(0)}$$
$$\hat{\boldsymbol{u}}_B^{(1)} = \boldsymbol{A}\boldsymbol{a}^{(0)} + \boldsymbol{D}\boldsymbol{d}^{(1)}$$
$$\hat{\boldsymbol{u}}_B^{(2)} = \boldsymbol{A}\boldsymbol{a}^{(1)} + \boldsymbol{D}\boldsymbol{d}^{(2)}$$
$$\cdots \tag{22}$$

In both solutions we have $\boldsymbol{d}^{(0)} = \boldsymbol{I}\boldsymbol{s}$ and $\boldsymbol{a}^{(0)} = \boldsymbol{P}\boldsymbol{U}\boldsymbol{I}\boldsymbol{s}$; $\boldsymbol{d}^{(1)} = (\boldsymbol{I} + \boldsymbol{R})\boldsymbol{s} = \boldsymbol{d}^{(0)} + \boldsymbol{R}\boldsymbol{s}$ and $\boldsymbol{a}^{(1)} = \boldsymbol{P}\boldsymbol{U}\boldsymbol{d}^{(1)}$; $\boldsymbol{d}^{(2)} = (\boldsymbol{I} + \boldsymbol{R} + \boldsymbol{R}^2)\boldsymbol{s} = \boldsymbol{d}^{(1)} + \boldsymbol{R}^2\boldsymbol{s}$ and $\boldsymbol{a}^{(2)} = \boldsymbol{P}\boldsymbol{U}\boldsymbol{d}^{(2)}$; and so on.

In the zeroth-order approximations, the solution $\hat{\boldsymbol{u}}_B^{(0)}$ contains only the term of $\boldsymbol{d}^{(0)}$, which represents the source waves departing in all members from all loaded joints. This means that the solution $\hat{\boldsymbol{u}}^{(0)} = \hat{\boldsymbol{u}}_B^{(0)} = \boldsymbol{D}\boldsymbol{d}^{(0)}$ satisfies the boundary conditions of the source joints, which are the zeroth-order approximation of the joint conditions for the entire structure, that is, $\hat{\boldsymbol{u}}_B^{(0)}$ satisfies the joint conditions at the source points only. On the other hand, the solution $\hat{\boldsymbol{u}}_A^{(0)}$ in Eq. (21) contains both terms of $\boldsymbol{d}^{(0)}$ and $\boldsymbol{a}^{(0)}$, the latter representing the arrival of the source waves at the neighboring joints. The sum of these two terms does not satisfy the same zeroth-order joint condition. Besides, $\boldsymbol{d}^{(0)}$ and $\boldsymbol{a}^{(0)}$ proceed simultaneously, which violates the condition of causality. The error of $\hat{\boldsymbol{u}}_A^{(0)}$ could, however, be compensated by adding the term of $\boldsymbol{D}\boldsymbol{R}\boldsymbol{s}$, which is a part of the next order approximation [$\boldsymbol{D}\boldsymbol{R}\boldsymbol{s} = \boldsymbol{D}(\boldsymbol{d}^{(1)} - d^{(0)})$].

In the first-order approximations the solution $\hat{\boldsymbol{u}}_B^{(1)}$ contains the terms $\boldsymbol{d}^{(1)}$ and $\boldsymbol{a}^{(0)}$. At a given time the arrival of the $\boldsymbol{a}^{(0)}$ wave is accompanied by the part of $\boldsymbol{R}\boldsymbol{s}$ in $\boldsymbol{d}^{(1)}$, and together they satisfy the boundary condition at the neighboring joints. The sum of two terms satisfies the first-order joint conditions, that is, the boundary conditions at the source joints and one ring of neighboring joints surrounding the source. However, the solution A of $\hat{\boldsymbol{u}}_A^{(1)}$ in Eq. (21) does not satisfy either one.

Continuing to the second-order approximations, we find that $\hat{\boldsymbol{u}}_B^{(2)}$ satisfies the second-order joint conditions (the conditions at the source joints and two rings of neighboring joints). The solution $\hat{\boldsymbol{u}}_A^{(2)}$ contains the extra part of $\boldsymbol{P}\boldsymbol{U}\boldsymbol{R}^2\boldsymbol{s}$ in $\boldsymbol{a}^{(2)}$, which disturbs the balance of boundary condition.

Next, we compare the Nth-order approximations, the solution $\hat{\boldsymbol{u}}_B^{(N)}$ in Eq. (15) satisfies the joint conditions at N rings of joints surrounding the source points (the Nth-order joint conditions) and fulfills the condition of causality, but the solution $\hat{\boldsymbol{u}}_A^{(N)}$ does not. The difference between these two solutions is given by the following formula:

$$\varepsilon(N) = \hat{\boldsymbol{u}}_A^{(N)} - \hat{\boldsymbol{u}}_B^{(N)} = \boldsymbol{A}\boldsymbol{a}_A - \boldsymbol{A}\boldsymbol{a}_B$$
$$= \boldsymbol{A}\left[\boldsymbol{a}^{(N)} - \boldsymbol{a}^{(N-1)}\right] = \boldsymbol{A}\boldsymbol{P}\boldsymbol{U}\boldsymbol{R}^N\boldsymbol{s} \tag{23}$$

Because the norm of $\boldsymbol{R}^N$ oscillates about a constant value as N increases, the value of each element in column $\boldsymbol{R}^N\boldsymbol{s}$ is always less than that of the corresponding element in $\boldsymbol{d}^{(N-1)} = (\boldsymbol{I} + \boldsymbol{R} + \boldsymbol{R}^2 + \cdots \boldsymbol{R}^{N-1})\boldsymbol{s}$. As a result, the ratio of each element in $\varepsilon(N)$ to the corresponding element of $\boldsymbol{d}^{(N-1)}$ decreases rapidly as N increases. Therefore, the difference between these two solutions is very small for large order of N. Hence, the final choice should be based on the accuracy of the lower-order approximation.

The lower-order ($N < n$) solutions $\hat{\boldsymbol{u}}_B^{(N)}(\boldsymbol{x}, \omega)$ are very crude approximations for steady-state waves. The corresponding transient waves solutions $\boldsymbol{u}_B^{(N)}(\boldsymbol{x}, t)$ as derived in Eq. (4) are, however, very accurate for the early time responses of the structure. This is seen by noting that the order N corresponds to the number of joints that have been passed by the source wave, and one can estimate the time of first arrival t_N of the fastest wave along the shortest path from the source to a receiver. For loadings with causal time function of $f(t)$, the transient wave $\boldsymbol{u}_B^{(N)}(\boldsymbol{x}, t)$ is then the exact solution of the structure from $t = 0$ to t_N. This is precisely the reason for expanding $(\boldsymbol{I} - \boldsymbol{R})^{-1}$ into power series within the framework of the theory of generalized rays.[4]

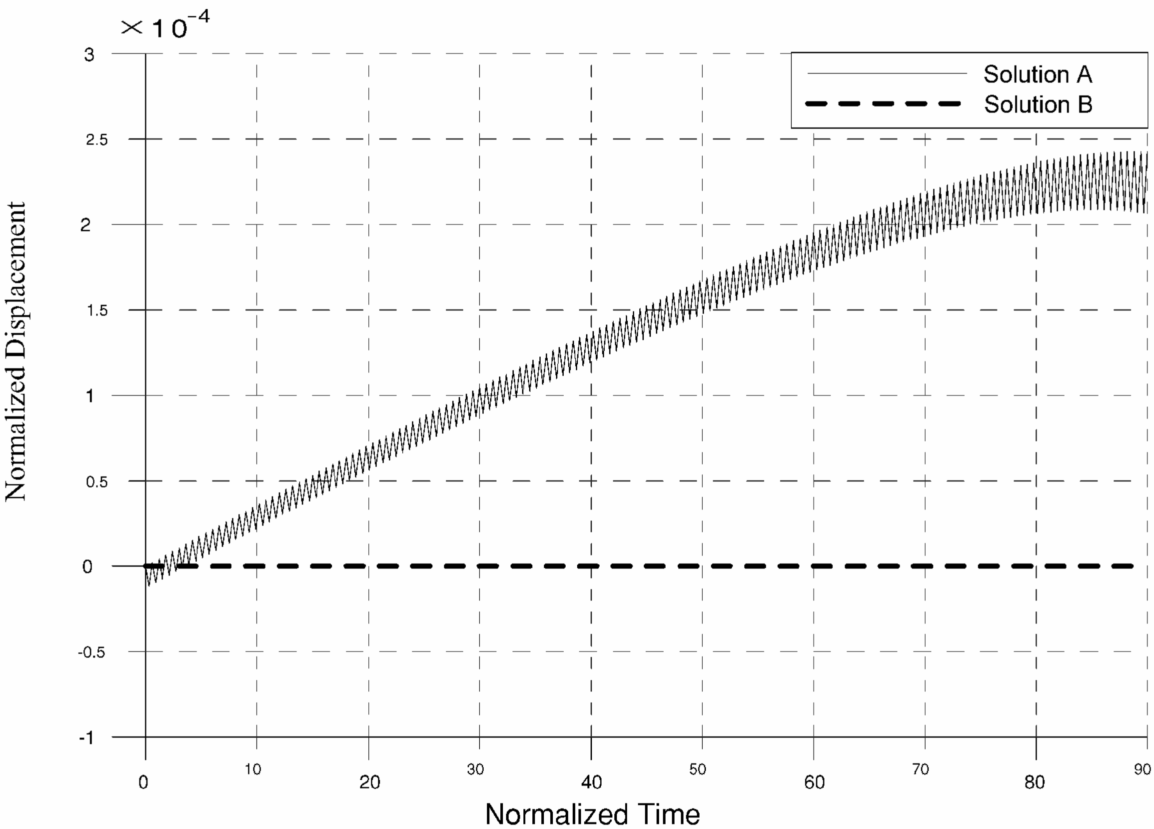

Fig. 2 Vertical displacement of joint 1 as calculated from axial displacement of member 1-2 by two procedures: ——, solution A; and - - - -, solution B.

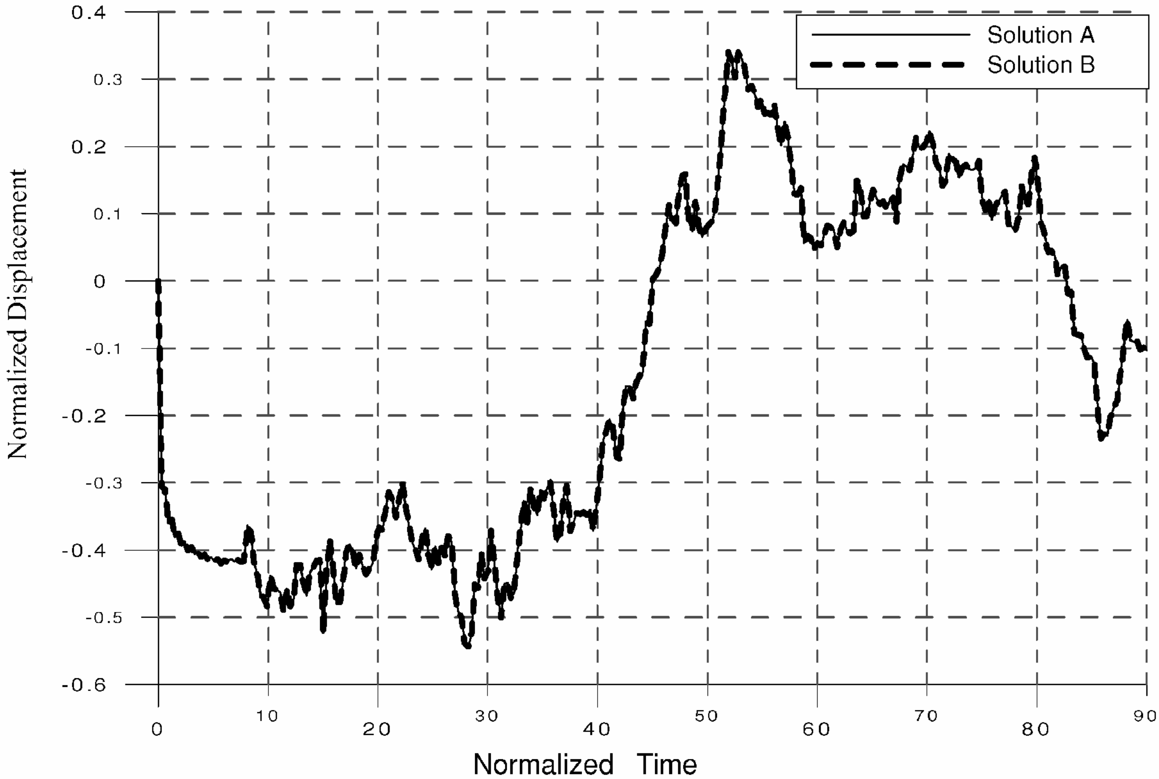

Fig. 3 Vertical displacement of joint 6 as calculated from axial displacement of member 6-5 by two procedures: ——, solution *A*; and - - - -, solution *B*.

On the basis of the new findings, we have recalculated the dynamic strains of the planar structure in Ref. 2 for $N = 30$ and find that there are indeed differences between the two solutions. The numerical differences, however, are so small that they could not be shown in the scales of Figs. 2–10 of Ref. 2.

We have also calculated anew the deflections at several joints of the same structure shown in Fig. 1 by both procedures. A unit impulsive force (delta function in time) is applied at joint 6, and the vertical displacement at joints 1 and 6 is shown in Figs. 2 and 3, respectively, with a solid line for $u_A^{(N)}(t)$ and a dashed line for $u_B^{(N)}(t)$. The difference between solution A and solution B ($N = 30$) is noticeable in Fig. 2, but the numerical values of both are nearly zero, which is the exact answer at this joint. The values for solution A (solid line) are less than 3×10^{-4}, and those for solution B (dashed line) are of the order 10^{-25}. In Fig. 3 these two solutions differ so little that the two lines coincide for the scale chosen.

V. Conclusions

In conclusion, we have found in this paper another set of approximate solutions for the arrival and departure wave vectors $[a^{(N-1)}, \boldsymbol{d}^{(N)}]$ in Eqs. (10) and (14), which is based on Eq. (5) of joint conditions. The arrival part, however, differs from the preceding approximation $[\boldsymbol{a}^{(N)}, \boldsymbol{d}^{(N)}]$ in Eqs. (10) and (11) as done in Ref. 2, which is based on Eq. (6) of phase relation. In addition, the phase relation is shown to be a condition of causality for $\tilde{\boldsymbol{d}}$ and $\boldsymbol{a}$ of two local coordinates, and $\boldsymbol{d}$ (cause) must precede $\boldsymbol{a}$ (effect) in the same member.

Neither set of approximations satisfies both Eqs. (5) and (6) concurrently, but the set $[\boldsymbol{a}^{(N-1)}, \boldsymbol{d}^{(N)}]$ of solution B satisfies the joint conditions approximately in successive orders as well as the condition of causality. The solution B found in this paper is thus preferred to the preceding one. The difference between these two solutions is given by Eq. (23), and the numerical value decreases rapidly as the order of N increases. Our conclusions are substantiated by the numerical results discussed in the preceding section.

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